

AN INTEGRAL LIFT OF THE Γ -GENUS

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ABSTRACT. The Hirzebruch genus of complex-oriented manifolds associated to Euler's Γ -function lifts to a homomorphism of ring-spectra associated to a family of deformations of the Dirac operator, parametrized by the homogeneous space Sp/U .

Introduction

Kontsevich, in his early work on deformation quantization [12 §4.6], drew attention to interesting formal properties of Euler's Γ -function, regarded as defining something like a Hirzebruch genus. This note presents that idea in the language of cobordism and formal groups, following [16]. The formalism of multiplicative power series defines a homomorphism

$$\chi_\infty : MU_* \rightarrow \mathbb{C}[v]$$

(of graded rings, with a book-keeping indeterminate v) having no very immediate integrality properties, but classical function theory [§2.3.1] shows it to take values in the ring $\mathbb{Q}[\tilde{\zeta}(\text{odd})]$ generated over the rationals by normalized zeta-values, usually expected to be transcendental. The principal result here [§3.1] is that a topologically reasonable homomorphism

$$MU \xrightarrow{\Gamma} MU \wedge_{M\mathrm{Sp}} \mathrm{KO} \xrightarrow{\cong [\frac{1}{2}]} \mathrm{Sp}/\mathrm{U} \wedge \mathrm{KO}[\frac{1}{2}]$$

of ring-spectra provides a lift of χ_∞ , via the composition

$$\mathrm{KO}_*(\mathrm{Sp}/\mathrm{U}) \xrightarrow{ch} H_*(\mathrm{Sp}/\mathrm{U}, \mathbb{Q}[v^{\pm 1}]) \longrightarrow H_*(\mathrm{BU}, \mathbb{Q}[v^{\pm 1}]) \longrightarrow \mathbb{C}[v^{\pm 1}]$$

which sends primitive generators of $H_*(\mathrm{Sp}/\mathrm{U}, \mathbb{Q})$ to odd ζ -values.

It was the appearance of these periods (and their relation to the theory of mixed Tate motives in algebraic geometry) that precipitated much of the interest in the Γ -genus. They appear in the lift as generic parameters for a family of deformations of a Dirac operator over the homogeneous space Sp/U . This seems to have interesting connections with [10] and [17].

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1. COIGNS OF VANTAGE

1.0 It's useful to distinguish a coordinate z at a point x_0 of a space X from the corresponding parametrization of a neighborhood $U \ni x_0$: the former is a nice function

$$X \supset U \xrightarrow{z} A$$

sending x_0 to 0 in some commutative ring A , while the latter is the map

$$\mathbf{z} : \text{Spec } A \rightarrow U \subset X$$

it defines (assuming we're in a context where this makes sense).

1.1 For example, at the point $x_0 = [1 : 1]$ of the projective line, we have a coordinate

$$[u : 1] \mapsto u - 1 := z$$

which defines the parametrization

$$z \mapsto [1 + z : 1]$$

of a neighborhood of $[1 : 1]$. Similarly,

$$[q : 1] \mapsto q^{-1} := z$$

is a coordinate at $[1 : 0] = \infty \in P_1$, while

$$[x : 1] \mapsto x := z$$

is a coordinate at $[0 : 1] = 0$.

1.2 An abelian group germ \mathbb{G} at $x_0 \in X$ is the germ of a function

$$\mathbb{G} : U \times U, x_0 \times x_0 \rightarrow U, x_0$$

satisfying identities such as

$$\mathbb{G}(x, \mathbb{G}(y, z)) = \mathbb{G}(\mathbb{G}(x, y), z), \quad \mathbb{G}(x, x_0) = \mathbb{G}(x_0, x) = x, \quad \&c ;$$

if \mathbb{G} is suitably analytic, then a coordinate z at x_0 associates to \mathbb{G} , the formal group law

$$(z \circ \mathbb{G})(\mathbf{z} \times \mathbf{z}) := z_0 +_{\mathbb{G}} z_1 \in A[[z_0, z_1]] .$$

For example, the additive group germ $\mathbb{G}_a(x, y) = x + y$ at $[0 : 1] \in P_1$ defines $z_0, z_1 \mapsto z_0 + z_1$, while the multiplicative group germ $\mathbb{G}_m(u, v) = uv$ at $[1 : 1]$ defines

$$z_0 +_{\mathbb{G}_m} z_1 = z_0 + z_1 + z_0 z_1$$

(with coordinates as above). Different choices of coordinate (for fixed \mathbb{G} and x_0) define, in general, distinct (but isomorphic) formal group laws: for example, if $t \in A^\times$ then $z = t^{-1}(u - 1)$ associates the formal group law

$$z_0, z_1 \mapsto z_0 + z_1 + tz_0z_1 .$$

to the multiplicative group at $[1 : 1]$.

1.3.1 The introduction of such a variable t suggests the consideration of families, or deformations, of group laws:

$$u, v \mapsto \frac{uv}{1 - t(u - 1)(v - 1)}$$

at $[1 : 1]$ (easily checked, eg for nilpotent t , to satisfy the axioms) is an interesting example. With coordinate as above, the associated group law

$$z_0, z_1 \mapsto \frac{z_0 + z_1 + (1 + t)z_0z_1}{1 - tz_0z_1} ;$$

is (strictly) isomorphic to $+\mathbb{G}_m$, under the coordinate change

$$z \rightarrow (1 + t)^{-1} \log \begin{bmatrix} t & 1 \\ -1 & 1 \end{bmatrix} (z) \in \mathbb{Q}[t][[z]] ;$$

note that the fractional linear transformation fixes $[1 : 1]$.

1.3.2 Similarly, $\exp_A(z) := 2 \sinh z/2$ defines

$$z_0 +_A z_1 = z_0(1 + \frac{1}{4}z_1^2)^{1/2} + z_1(1 + \frac{1}{4}z_0^2)^{1/2} \in \mathbb{Z}[\frac{1}{2}][[z_0, z_1]] ,$$

which is a specialization (at $\delta = -\frac{1}{8}$, $\epsilon = 0$) of the formal group law

$$z_0 +_E z_1 = \frac{z_0 R(z_1) + z_1 R(z_0)}{1 - \epsilon z_0^2 z_1^2}$$

defined by Jacobi's quartic $Y^2 = R(X)^2 := 1 - 2\delta X^2 + \epsilon X^4$.

1.4 The focus of this note is the group germ

$$\mathbb{G}_\infty : [q_0 : 1], [q_1 : 1] \mapsto [\Gamma(\log_\infty(q_0^{-1}) + \log_\infty(q_1^{-1})) : 1]$$

at $\infty \in P_1(\mathbb{R})$ defined by the expansion

$$\exp_\infty(z) := z \exp(\gamma z - \sum_{k \geq 2} \frac{\zeta(k)}{k} (-z)^k) \in \mathbb{R}[[z]]$$

of the entire function $\Gamma(z)^{-1}$ near 0 (with $\log_\infty(z)$ denoting its formal composition inverse): thus

$$z_0 +_{\mathbb{G}_\infty} z_1 = \Gamma(\log_\infty(z_0) + \log_\infty(z_1))^{-1} = z_0 + z_1 + 2\gamma z_0 z_1 + \cdots \in \mathbb{R}[[z_0, z_1]]$$

with $z_k = q_k^{-1}$. Ohm's law for parallel resistors¹, in comparison, defines a group germ

$$[q_0 : 1], [q_1 : 1] \mapsto [1 : q_0^{-1} + q_1^{-1}]$$

¹a.k.a. the harmonic mean of Archytas of Tarentum

at ∞ , which (because $\frac{xy}{x+y}$ is not differentiable at $(0,0)$) is not analytic.

2. CHARACTERISTIC CLASSES AND HIRZEBRUCH GENERA

2.1 A complex line bundle $\lambda \in H^1(X, \mathbb{C}^\times)$ has an associated class

$$\lambda^{-1}d\lambda \mapsto 2\pi i[\lambda] : H^1(X, \mathbb{Z}(1)) \rightarrow H^2(X, 2\pi i\mathbb{Z})$$

corresponding to the coordinate [1 §2.3, 19 §5.10]

$$z = vx \in H^{\text{even}}(X, \mathbb{Z}[v^{\pm 1}])$$

on the Picard group of topological complex line bundles. Interpreting v as the product of the Bott class with Deligne's motive $2\pi i$ reconciles some conventions of algebraic geometry with those of algebraic topology: for example

$$\frac{\pi[\lambda]}{\sin \pi[\lambda]} \mapsto \frac{vx/2}{\sinh vx/2}.$$

When the grading is of background interest, I'll set v equal to 1.

2.2.1 A (one-dimensional) formal group law over a \mathbb{Q} -algebra A can be written uniquely as

$$z_0 +_{\mathbb{G}} z_1 = \exp_{\mathbb{G}}(\log_{\mathbb{G}}(z_0) + \log_{\mathbb{G}}(z_1)) ;$$

in that case let

$$H_{\mathbb{G}}(z) := \frac{z}{\exp_{\mathbb{G}}(z)} \in A[[z]]^\times$$

denote its Hirzebruch multiplicative series [8 §15.5]. The function

$$M \mapsto \left(\prod_{i=1}^{i=n} H_{\mathbb{G}}(vx_i) \right) [M] \in A[v]$$

from (cobordism classes of) compact closed complex-oriented manifolds of real dimension $2n$, with Chern roots x_i , defines a homomorphism

$$\chi_{\mathbb{G}} : MU_* \rightarrow A[v]$$

of graded rings: the Hirzebruch genus associated to the group law \mathbb{G} . By a theorem of Mishchenko,

$$\log_{\mathbb{G}}(v) = \sum_{n \geq 1} \frac{\chi_{\mathbb{G}}(P_{n-1}(\mathbb{C}))}{n} \in A[[v]] ;$$

the deformation of the multiplicative group in §1.3.1, for instance, represents Hirzebruch's genus χ_{-t} genus (defined on smooth projective complex varieties by

$$V \mapsto \sum (-1)^p (-t)^q \dim_{\mathbb{C}} H_{\text{dg}}^{p,q}(V) v^{\dim_{\mathbb{C}} V}.$$

The coordinate rescaling $v \mapsto t^{-1/2}v$ sends its logarithm to

$$\sum_{n \geq 1} [n](t) \frac{v^n}{n}$$

(with Gaussian $\frac{t^{n/2} - t^{-n/2}}{t^{1/2} - t^{-1/2}} = [n](t)$), and its formal group law to

$$X, Y \mapsto \frac{X + Y + (t^{1/2} + t^{-1/2})vXY}{1 - vXY}$$

(which is symmetric under the involution $t \mapsto 1/t$).

2.2.2 I'll refer below to MSO , MU , and $M\mathbb{S}p$ as the cobordism theories of \mathbb{R} , \mathbb{C} , and \mathbb{H} -oriented manifolds, respectively.

The Pontryagin classes

$$p_t^{\text{SO}}(V) = \sum_{k \geq 0} p_k^{\text{SO}}(V) t^{2k} := \sum_{k \geq 0} (-1)^k c_{2k}(V \otimes \mathbb{C}) t^{2k}$$

of a real vector bundle V are defined in terms of the Chern classes of its complexification; if V was complex to begin with, then

$$c_t(V \otimes \mathbb{C}) = \sum_{k \geq 0} c_k(V \otimes \mathbb{C}) t^k = c_t(V) \cdot c_t(\overline{V})$$

equals

$$\prod (1 - x_i^2 t^2) = \sum (-1)^k e_k(x_i^2) t^{2k},$$

which expresses the Pontryagin classes

$$p_k^{\text{SO}}(V) = e_k(x_i^2)$$

in terms of elementary symmetric functions of the Chern roots x_i of $V \otimes \mathbb{C}$.

If $H_{\mathbb{G}}(z) := \hat{H}_{\mathbb{G}}(z^2)$ is an even power series, then the associated genus $\chi_{\mathbb{G}}$ of a \mathbb{C} -oriented manifold M can be evaluated in terms of Pontryagin classes, since

$$\prod \hat{H}_{\mathbb{G}}(x_i^2) := \mathbf{H}_{\mathbb{G}}(p_k^{\text{SO}})$$

for some polynomial $\mathbf{H}_{\mathbb{G}}$; this factors $\chi_{\mathbb{G}}$ through a homomorphism

$$MU \longrightarrow MSO \xrightarrow{\hat{\chi}_{\mathbb{G}}} A[v] .$$

The complex vector bundle underlying a quaternionic vector bundle V , on the other hand, can be decomposed as the sum of a complex bundle with its conjugate. In that case we have

$$p_t^{\text{SO}}(V) = p_t^{\text{SO}}(W \oplus \overline{W}) = p_t^{\text{SO}}(W)^2$$

(at least, with coefficients in a $\mathbb{Z}[\frac{1}{2}]$ -algebra). The symplectic Pontryagin classes of V are defined by

$$p_t^{\text{Sp}}(V) = \sum (-1)^k c_{2k}(V) t^{2k}$$

[20], so $p_t^{\text{Sp}}(V) = p_t^{\text{SO}}(W)$, hence $p_t^{\text{SO}}(V) = (p_t^{\text{Sp}}(W))^2$. Since $p_t^{\text{SO}}(V)$ can be expressed in terms of the power sums $\sum x_i^{2k} = s_k^{\text{SO}}$ of the Chern roots of $V \otimes \mathbb{C}$ as

$$\exp\left(\sum s_{2k}^{\text{SO}} \frac{t^{2k}}{k}\right),$$

we have

$$s_{2k}^{\text{SO}} := s_{2k}(V \otimes \mathbb{C}) = 2s_{2k}(V) := 2s_{2k}^{\text{Sp}}$$

(in terms of the Chern roots of the complex structure underlying a quaternionic structure on V).

2.3.1 Rewriting the logarithm of Weierstrass's product formula for Γ , we have

$$\Gamma(1+z) = \exp(-\gamma z + \sum_{k>1} \frac{\zeta(k)}{k} (-z)^k);$$

from this, and the duplication formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

it follows that

$$\frac{x/2}{\sinh x/2} = \exp\left(\sum_{k \geq 1} \frac{\zeta(2k)}{(2\pi i)^{2k}} \frac{x^{2k}}{2k}\right),$$

with rational coefficients

$$\frac{\zeta(2k)}{(2\pi i)^{2k}} = -\frac{B_{2k}}{2(2k)!}.$$

The \hat{A} -genus of an oriented manifold (corresponding to the group law in §1.3.2) can thus be calculated by evaluating

$$\prod \left(\frac{v x_i / 2}{\sinh v x_i / 2} \right) = \exp\left(-\sum \frac{B_{2k}}{(2k)!} \frac{s_{2k}^{\text{SO}}}{4k} v^{2k}\right)$$

on its fundamental class. If the manifold is \mathbb{H} -oriented, this characteristic class equals the product

$$\prod \left(\frac{x_i / 2}{\sinh x_i / 2} \right)^{1/2}$$

(now taken over the Chern roots of the complex bundle underlying the \mathbb{H} -oriented structure).

Proposition. *The genus of complex-oriented manifolds defined by the multiplicative series*

$$H_{\mathbb{G}_\infty}(x) = \Gamma(1+[\lambda]) = \left(\frac{x/2}{\sinh x/2} \right)^{1/2} \exp\left(i \frac{\gamma}{2\pi} x + \sum \frac{\zeta(\text{odd})}{(2\pi i)^{\text{odd}}} \frac{x^{\text{odd}}}{\text{odd}}\right) \in \mathbb{C}[[x]]$$

agrees on the image of $M\text{Sp}$ in $M\text{U}$ with the \hat{A} -genus.

[Because the odd terms in the exponential cancel, for a bundle of the form $W \oplus \overline{W}$.]

2.3.2 Note that the Witten genus [14]

$$H_W(x) = \frac{x/2}{\sinh x/2} \prod_{n \geq 1} [(1 - q^n e^x)(1 - q^n e^{-x})]^{-1}$$

can be written similarly, in terms of Eisenstein series, as

$$\exp\left(\sum_k G_{2k}(q) \frac{x^{2k}}{2k}\right);$$

but this deformation of the \hat{A} -genus is an **even** function of x .

2.4 The elementary symmetric functions e_n and the corresponding power sums \mathbf{s}_n are related by

$$e(z) = \sum_{n \geq 0} e_n z^n := \prod_{k \geq 1} (1 + x_k z) = \exp\left(-\sum_{n \geq 1} \frac{\mathbf{s}_n}{n} (-z)^n\right).$$

The assignment $x_k \mapsto 1/k$ [5, 9, 14 I §2 ex 21] requires some care, but, suitably interpreted, sends \mathbf{s}_k to $\zeta(k)$ if $k > 1$, and \mathbf{s}_1 to γ . The formal power series

$$\text{Exp}_\infty(z) = z \cdot e(z)$$

thus specializes to $\exp_\infty(z)$ under this mapping, defining a lift \mathbf{G}_∞ of \mathbb{G}_∞ to a formal group law over the polynomial algebra $\mathbb{Z}[e_n \mid n \geq 1]$. Since its exponential is defined over \mathbb{Z} , it is of additive type, and is in fact the universal such formal group law.

Similarly

$$H_{\mathbf{G}_\infty}(z) = \sum_{k \geq 0} h_k (-z)^k,$$

in terms of the complete symmetric functions h_k .

3. THE REAL STRUCTURE OF MU

3.1 Proposition. *In the homotopy-commutative diagram*

$$\begin{array}{ccccc} MU & \xrightarrow{\quad} & S^0[BU_+] \wedge \mathbb{H}\mathbb{Z} & \longrightarrow & S^0[\text{Sp}/U_+ \wedge B\text{Sp}_+] \wedge \mathbb{H}\mathbb{Z}[\tfrac{1}{2}] \\ & \searrow \Gamma[\tfrac{1}{2}] & & & \downarrow \zeta(\text{even}) \\ S^0[\text{Sp}/U_+] \wedge M\text{Sp} & \longrightarrow & S^0[\text{Sp}/U_+] \wedge \text{KO}[\tfrac{1}{2}] & \longrightarrow & S^0[\text{Sp}/U_+] \wedge \text{H}\mathbb{Q}[v^{\pm 1}] \\ & & \uparrow & \searrow & \downarrow \zeta(\text{odd}) \\ M\text{Sp} & \xrightarrow{\hat{A}} & \text{KO} & \longrightarrow & \text{HC}[v^{\pm 1}] \end{array}$$

of spectra, the diagonal composition represents the Γ -genus.

3.2 Proof. Here $S^0[G_+]$ is the suspension ring-spectrum defined by an H -space G , such as the fiber Sp/U ($\sim \Omega\mathrm{Sp} \sim B(\mathrm{U}/\mathrm{O})$) of the quaternionification map $BU \rightarrow B\mathrm{Sp}$. Note that the inclusion of the fiber into BU makes $S^0[BU_+]$ (and hence MU) into $S^0[\mathrm{Sp}/\mathrm{U}_+]$ -modules).

The two vertical maps at the lower left side of the diagram are the obvious smash products with the unit $S^0 \rightarrow S^0[\mathrm{Sp}/\mathrm{U}_+]$, while the horizontal maps across the middle of the diagram are smash products with the \hat{A} -genus, regarded as defined by the index of a Dirac operator on an \mathbb{H} -oriented manifold, followed by the Chern character on KO . The top left-hand map is just the total characteristic number homomorphisms of Boardman and Quillen, and can alternately be described as the composition

$$MU_* \rightarrow MU_* \otimes S_* \rightarrow \mathbb{Z} \otimes S_* = S_*$$

of the total Landweber-Novikov operation with Steenrod's cycle map

$$1 \in H^0(BU, \mathbb{Z}) \rightarrow H^0(MU, \mathbb{Z}) = [MU, \mathbb{H}\mathbb{Z}]_0 .$$

The (related) upper left-hand vertical and upper right-hand horizontal maps are more interesting. An element of $M\mathrm{Sp}_*(\mathrm{Sp}/\mathrm{U}_+)$ can be interpreted as the bordism class of an \mathbb{H} -oriented manifold M , equipped with a map to Sp/U , and if we regard M as merely complex-oriented, then the product composition

$$M \rightarrow \mathrm{Sp}/\mathrm{U}_+ \wedge BU_+ \rightarrow BU_+$$

defines a new complex orientation on M , and thus a ring homomorphism

$$M\mathrm{Sp}_*(\mathrm{Sp}/\mathrm{U}_+) \rightarrow MU_* .$$

By [3], this is in fact an isomorphism away from the prime (2); similarly, the composition

$$\mathrm{Sp}/\mathrm{U}_+ \wedge B\mathrm{Sp}_+ \rightarrow \mathrm{Sp}/\mathrm{U}_+ \wedge BU_+ \rightarrow BU_+$$

defines an isomorphism

$$H_*(\mathrm{Sp}/\mathrm{U}, \mathbb{Z}[\frac{1}{2}]) \otimes_{\mathbb{Z}[\frac{1}{2}]} H_*(B\mathrm{Sp}, \mathbb{Z}[\frac{1}{2}]) \cong H_*(BU, \mathbb{Z}[\frac{1}{2}])$$

of Hopf algebras, which is the upper right-hand map.

Since the diagonal maps are defined by the diagram, only the right-hand vertical maps remain to be constructed, but that is the content of §2.4: the power-sum generators of $H_*(BU, \mathbb{Q})$ map to normalized zeta-values

$$s_k \mapsto \tilde{\zeta}(k) := (2\pi i)^{-k} \zeta(k) \text{ if } k > 1 \quad , \quad \mapsto -\frac{\gamma}{2\pi} \cdot i \text{ if } k = 1 .$$

This is factored into two steps:

$$\zeta(\text{even}) : s_{2k} \mapsto \frac{B_{2k}}{4k(2k)!} \in \mathbb{Q}$$

can be interpreted as defining the \hat{A} -genus, while

$$\zeta(\text{odd}) : s_{2k+1} \mapsto (-1)^{k+1} (2\pi)^{-2k-1} \zeta(2k+1) \cdot i .$$

3.3 Complex conjugation on MU is represented by the coordinate change $z \mapsto [-1](z)$ on the formal group, which corresponds to complex conjugation on the value group of the Γ -genus. In other words, the Γ -genus is naturally \mathbb{Z}_2 -equivariant, with respect to the Galois action defined by the Real structure on complex cobordism.

Away from (2), the Landweber-Novikov algebra of cobordism operations is an enveloping algebra of a \mathbb{Z}_2 -graded Lie (NB not super-Lie) algebra. The odd part corresponds, in classical Lie theory, to the tangent space of the symmetric space associated to the complexification of a real Lie group; it acts transitively on $\text{Spec } H_*(\text{Sp}/\text{U}, \mathbb{Q})$, cf. [3, 17].

4. CLOSING REMARKS

4.1 The index map $M\text{Sp} \rightarrow \text{KO}$ dates back to Conner and Floyd's 1968 work on the relation of cobordism to K -theory, but seems to have received remarkably little attention: it is surely represented geometrically by a Dirac operator on \mathbb{H} -oriented manifolds, but the question of a nice construction seems not to have caught the differential geometers' attention. In view of this, I have not tried to define an explicit family of deformations of such an operator over Sp/U .

4.2 R. Lu [8] has proposed an analytic interpretation of a variant of the Γ -genus of a complex-oriented M as a \mathbb{T} -equivariant Euler class of its free loop space, following Atiyah ([2]; see also [1]). Lu's construction depends on a choice of **polarization**

$$\begin{array}{ccc} \text{U/O} & \longrightarrow & B\text{Gl}_{\text{res}} \sim B(\mathbb{Z} \times \text{BO}) \\ \uparrow \text{dotted} & \nearrow & \downarrow \\ LM & \longrightarrow & L\text{BU} \sim B(L\text{U}) \sim B(\mathbb{Z} \times \text{BU}) \end{array}$$

of the tangent bundle of LM : that is, a lift of the map classifying its tangent bundle, to the restricted Grassmannian defined by writing loops in the tangent space as a sum of something like positive and negative-frequency components. Since M is complex-oriented, such a lift exists, but is not in general unique: it can be twisted by a map

$$LM \rightarrow \text{U/O} \sim \Omega(\text{Sp}/\text{U})$$

[6 §2, 7, 10]. The free loops on a map $\alpha : M \rightarrow \text{Sp}/\text{U} \in \text{KO}^2(X)$ thus define a twist

$$L(\alpha) : LM \rightarrow L(\text{Sp}/\text{U}) \sim \text{U/O} \times \text{Sp}/\text{U} \rightarrow \text{U/O} ;$$

its restriction to the subspace M of constant loops defines a map to $\text{Sp}/\text{U} \sim \Omega\text{Sp}$ which acts naturally on $\text{U/O} \sim \Omega(\text{Sp}/\text{U})$, and it seems reasonable to expect that Lu's class for the polarized manifold $(M, L(\alpha))$ can be expressed

in terms of $\Gamma(M)$ evaluated at suitable values $s_{2k}(\alpha)$ of the deformation parameters.

4.3 I have also not tried to pin down the two-local properties of Γ , which seem quite interesting. Away from (2), Sp/U is closely related [4] to $B\mathrm{Sp}(\mathbb{Z})^+$, which is in turn related (via Siegel) to the K -theory spectrum of the symmetric monoidal category of Abelian varieties. This suggests that one might hope to see in the Γ -genus, some homotopy-theoretic residue of the intermediate Jacobians of complex projective manifolds.

4.4 Kontsevich's original remarks were motivated by questions of quantization, and nothing in the discussion above says much about that: homotopy theory is often revealing about the bones of a subject, without resolving the surrounding analytical structures.

It is intriguing that the points 0, 1, ∞ on the projective line seem to have naturally associated genera and cohomology theories: the additive group at zero is related to de Rham theory, and the multiplicative group at one to K -theory. The association of the point at infinity with the Kontsevich genus suggests it might be related to a Galois theory of asymptotic expansions, along lines suggested by Cartier, Connes, Kreimer, Marcolli, and others.

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